

SUBAMENABLE G -SPACES AND THEIR PARTITIONS

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ABSTRACT. A group G is called *subamenable* if there is a left-invariant submeasure $\mu : \mathcal{P}(G) \rightarrow [0, 1]$ such that for each $\varepsilon > 0$ and each subset $A \subset G$ with $\mu(A) < 1$ there is a set $B \subset G \setminus A$ such that $\mu(B) < \varepsilon$ and $FB = G$ for some finite subset $F \subset G$. The class of subamenable groups is quite wide: it contains (i) all groups admitting a homomorphism onto a subamenable group; (ii) all infinite groups admitting a totally bounded group topology, (iii) all groups containing a normal infinite amenable subgroup. We do not know if each infinite group is subamenable. We prove that for every $k \in \mathbb{N}$ a subamenable group G can be partitioned into two sets A, B such that for each k -element subset $K \subset G$ the sets KA and KB are not thick.

In this paper we continue the studies [14]–[19] of combinatorial properties of partitions of G -spaces and groups.

By a G -space we understand a non-empty set X endowed with a left action of a group G . The image of a point $x \in X$ under the action of an element $g \in G$ is denoted by gx . For two subsets $F \subset G$ and $A \subset X$ we put $FA = \{fa : f \in F, a \in A\} \subset X$.

1. PRETHICK SETS IN PARTITIONS OF G -SPACES

A subset A of a G -space X is called

- *large* if $FA = X$ for some finite subset $F \subset G$;
- *thick* if for each finite subset $F \subset G$ there is a point $x \in X$ with $Fx \subset A$;
- *prethick* if KA is thick for some finite set $K \subset G$.

Now we insert number parameters in these definitions. Let $k, m \in \mathbb{N}$. A subset A of a G -space X is called

- *m -large* if $FA = X$ for some subset $F \subset G$ of cardinality $|F| \leq m$;
- *m -thick* if for each finite subset $F \subset G$ of cardinality $|F| \leq m$ there is a point $x \in X$ with $Fx \subset A$;
- *(k, m) -prethick* if KA is m -thick for some set $K \subset G$ of cardinality $|K| \leq k$;
- *(k, ω) -prethick* if KA is thick for some set $K \subset G$ of cardinality $|K| \leq k$.

In the dynamical terminology [8, 4.38], large subsets are called syndetic and prethick subsets are called piecewise syndetic. We note also that these notions can be defined in much more general context of balleanes [18], [19].

The following proposition is well-known [8, 4.41], [16, 1.3], [18, 11.2].

Proposition 1.1. *For any finite partition $X = A_1 \cup \dots \cup A_n$ of a G -space X one of the cells A_i is prethick and hence (k, ω) -prethick for some $k \in \mathbb{N}$.*

For finite groups the number k in this proposition can be bounded from above by $n(\ln(\frac{|G|}{n}) + 1)$. We consider each group G as a G -space endowed the natural left action of G .

Proposition 1.2. *Let G be a finite group and $n, k \in \mathbb{N}$ be numbers such that $k \geq n \cdot (\ln(\frac{|G|}{n}) + 1)$. For any n -partition $G = A_1 \cup \dots \cup A_n$ of G one of the cells A_i is k -large and hence (k, ω) -prethick.*

Proof. One of the cells A_i of the partition has cardinality $|A_i| \geq \frac{|G|}{n}$. Then by [21] or [3, 3.2], there is a subset $B \subset G$ of cardinality $|B| \leq \frac{|G|}{|A_i|}(\ln |A_i| + 1) \leq n(\ln(\frac{|G|}{n}) + 1) \leq k$ such that $G = BA_i$. It follows that the set A_i is k -large and hence (k, ω) -prethick. \square

For G -spaces we have the following quantitative version of Proposition 1.1.

Proposition 1.3. *Let $m, n \in \mathbb{N}$. For any n -partition $X = A_1 \cup \dots \cup A_n$ of a G -space X one of the cells A_i is (m^{n-1}, m) -prethick in X .*

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Proof. For $n = 1$ the proposition is trivial. Assume that it has been proved for some n and take any partition $X = A_0 \cup \dots \cup A_n$ of X into $(n + 1)$ pieces. If the cell A_0 is $(1, m)$ -prethick, then we are done. If not, then there is a set $F \subset G$ of cardinality $|F| \leq m$ such that $Fx \not\subset A_0$ for all $x \in X$. This implies that $x \in F^{-1}(A_1 \cup \dots \cup A_n)$ and then by the inductive assumption, there is an index $1 \leq i \leq n$ such that the set $F^{-1}A_i$ is (m^{n-1}, m) -prethick. The latter means that there is a subset $E \subset G$ of cardinality $|E| \leq m^{n-1}$ such that $EF^{-1}A_i$ is m -thick. Since $|EF^{-1}| \leq |E| \cdot |F| \leq m^{n-1}m = m^n$, the set A_i is (m^n, m) -prethick. \square

Looking at Proposition 1.3 it is natural to ask what happens for $m = \omega$. Is there any hope to find for every $n \in \mathbb{N}$ a finite number k_n such that for each n -partition $X = A_1 \cup \dots \cup A_n$ some cell A_i of the partition is (k_n, ω) -prethick? In fact, G -spaces with this property do exist.

Example 1.4. Let X be an infinite set endowed with the natural action of the group $G = S_X$ of all bijections of X . Then each subset $A \subset X$ of cardinality $|A| = |X|$ is 2-large, which implies that for each finite partition $X = A_1 \cup \dots \cup A_n$ one of the cell A_i has cardinality $|A_i| = |X|$ and hence is 2-large and $(2, \omega)$ -prethick.

However the G -space X described in Example 1.4 is rather pathologic. In the next section we introduce a class of G -spaces (called subamenable) admitting for every $k \in \mathbb{N}$ a partition $X = A \cup B$ into two cells none of which is (k, ω) -prethick.

2. SUBAMENABLE G -SPACES AND THEIR PARTITIONS

A function $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ defined on the family of all subsets of a G -space X is called

- G -invariant if $\mu(gA) = \mu(A)$ for each $g \in G$ and a subset $A \subset X$;
- monotone if $\mu(A) \leq \mu(B)$ for any subsets $A \subset B \subset G$;
- subadditive if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any sets $A, B \subset X$;
- a submeasure if μ is monotone, subadditive, and $\mu(\emptyset) = 0$;
- syndetic if for each subset $A \subset X$ with $\mu(A) < 1$ and each $\varepsilon > 0$ there is a large subset $L \subset X \setminus A$ with $\mu(L) < \varepsilon$.

A G -space X is defined to be *subamenable* if X admits a syndetic G -invariant submeasure $\mu : \mathcal{P}(X) \rightarrow [0, 1]$. Such function μ will be called a *syndetic submeasure* on X .

Proposition 2.1. If $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ is a syndetic submeasure on a G -space X , then $\mu(F) = 0$ and $\mu(X \setminus F) = 1$ for every finite subset $F \subset X$. Consequently, X is infinite.

Proof. Fix any finite set $F \subset X$. Since μ is subadditive, the equality $\mu(F) = 0$ will follow as soon as we check that $\mu(\{x\}) = 0$ for each point $x \in F$ and each $\varepsilon > 0$. Since the empty set has submeasure $\mu(\emptyset) = 0 < 1$, there exists a large set $L \subset X \setminus \emptyset = X$ with submeasure $\mu(L) < \varepsilon$. Since L is large in X , there is a finite set $E \subset G$ such that $X = EL$. Then $gx \in L$ for some $g \in E^{-1}$ and by the G -invariance and the monotonicity of μ , we get $\mu(\{x\}) = \mu(\{gx\}) \leq \mu(L) < \varepsilon$. So, $\mu(F) = 0$.

Assuming that $\mu(X \setminus F) < 1$, we would conclude that the set $X \setminus (X \setminus F) = F$ is large in X . Then for some finite set $B \subset G$ we get $X = BF$, which implies that X is finite and hence $\mu(X) = 0 < 1$. In this case the definition of a syndetic submeasure μ guarantees that the complement $X \setminus X = \emptyset$ is large, which is a desired contradiction showing that $\mu(X \setminus F) = 1$. \square

For subamenable G -spaces we have the following result completing Propositions 1.1–1.3.

Theorem 2.2. Let G be a countable group and X be a subamenable G -space. Then for every $k \in \mathbb{N}$ there is a partition $X = A \cup B$ of X into two subsets none of which is (k, ω) -prethick.

Proof. The G -space X , being subamenable, carries a syndetic submeasure $\mu : \mathcal{P}(G) \rightarrow [0, 1]$. Fix any $k \in \mathbb{N}$ and choose an enumeration $(K_n)_{n=1}^\infty$ of all k -element subsets of G .

Using the definition of a syndetic submeasure, we can inductively construct two sequences $(A_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ of large subsets of X satisfying the following conditions for every $n \in \mathbb{N}$:

- (1) $A_n \subset X \setminus \bigcup_{i < n} K_n^{-1} K_i B_i$;
- (2) $\mu(A_n) < \frac{1}{k^{2 \cdot 2^n}}$;
- (3) $B_n \subset X \setminus \bigcup_{i \leq n} K_n^{-1} K_i A_i$;
- (4) $\mu(B_n) < \frac{1}{k^{2 \cdot 2^n}}$.

At each step the choice of the set A_n is possible as

$$\mu\left(\bigcup_{i < n} K_n^{-1} K_i B_i\right) \leq \sum_{i < n} \sum_{x \in K_n^{-1} K_i} \mu(x B_i) = \sum_{i < n} |K_n^{-1} K_i| \cdot \mu(B_i) \leq \sum_{i < n} k^2 \frac{1}{k^{2i}} < 1$$

by the subadditivity of μ . By the same reason, the set B_n can be chosen.

After completing the inductive construction, we get the disjoint sets $A = \bigcup_{n=1}^{\infty} K_n A_n$ and $B = \bigcup_{n=1}^{\infty} K_n B_n$.

It remains to check that the sets A and $X \setminus A$ are not (k, ω) -prethick. Given any k -element subset $K \subset G$ we need to prove that the sets KA and $K(X \setminus A)$ are not thick. Find $n \in \mathbb{N}$ such that $K_n = K^{-1}$.

Since the set $K_n B_n$ is disjoint with A , the large set B_n is disjoint with $K_n^{-1} A = KA$, which implies that $X \setminus KA$ is large and KA is not thick.

Next, we show that the set $K(X \setminus A) = K_n^{-1}(X \setminus A)$ is not thick. We claim that $A_n \subset X \setminus K_n^{-1}(X \setminus A)$. Assuming the converse, we can find a point $a \in A_n \cap K_n^{-1}(X \setminus A)$. Then $K_n a$ intersects $X \setminus A$, which is not possible as $K_n a \subset K_n A_n \subset A$. So, the set $X \setminus K(X \setminus A) \supset A_n$ is large, which implies that $K(X \setminus A)$ is not thick. \square

In light of Theorem 2.2 it is important to detect subamenable G -spaces. We give a measure-topological condition implying the subamenability.

A topology τ on a G -space will be called

- *G -invariant* if $gU \in \tau$ for each open set $U \in \tau$;
- *totally bounded* if each non-empty open set $U \in \tau$ is large;
- *regular and crowded* if the topological space (X, τ) is regular and has no isolated points.

Proposition 2.3. *A G -space X is subamenable if it admits a crowded regular totally bounded G -invariant topology τ and a G -invariant probability finitely additive measure μ defined on the algebra generated by closed subsets of (X, τ) .*

Proof. Define a syndetic submeasure $\bar{\mu} : \mathcal{P}(X) \rightarrow [0, 1]$ letting $\bar{\mu}(A) = \mu(\bar{A})$ where \bar{A} is the closure of A in the topology τ . The G -invariance of the measure μ and the topology τ implies the G -invariance of the submeasure $\bar{\mu}$. Since $\overline{A \cup B} = \bar{A} \cup \bar{B}$, the additivity of the measure μ implies the subadditivity of $\bar{\mu}$.

It remains to check that μ is syndetic. Given any subset $A \subset X$ with submeasure $\bar{\mu}(A) = \mu(\bar{A}) < 1$, we conclude that the set $U = X \setminus \bar{A}$ is not empty. Choose any natural number n with $\frac{1}{n} < \varepsilon$ and find non-empty open sets $U_1, \dots, U_n \subset U$ with pairwise disjoint closures (such sets exist since the topology τ is regular and crowded). Now the additivity of the measure μ guarantees that $\bar{\mu}(U_i) = \mu(\bar{U}_i) \leq \frac{1}{n} < \varepsilon$ for some i . Since the topology τ is totally bounded, the open set U_i is large. \square

3. SUBAMENABLE GROUPS

A group G is called *subamenable* if it is subamenable as a G -space endowed with the natural left action of the group G .

The following proposition shows that the class of subamenable groups is rather wide.

Theorem 3.1. *An infinite group G is subamenable if it satisfies one of the conditions:*

- (1) *for some subgroup $H \subset G$ the quotient G -space $G/H = \{gH : g \in G\}$ is subamenable;*
- (2) *for some normal subgroup $H \subset G$ the quotient group G/H is subamenable;*
- (3) *G admits a homomorphism onto an infinite totally bounded topological group;*
- (4) *G is residually finite;*
- (5) *G contains an infinite normal amenable subgroup.*

Proof. 1. Assume that for some subgroup $H \subset G$ the quotient space G/H is subamenable and let μ be a syndetic submeasure on the G -space $X = G/H$. Let $q : G \rightarrow G/H$, $q : g \mapsto gH$, be the quotient map and $\mu' : \mathcal{P}(G) \rightarrow [0, 1]$ be the submeasure defined by $\mu'(A) = \mu(q(A))$. To see that μ' is a syndetic submeasure on G , take any subset $A \subset G$ with $\mu'(A) = \mu(q(A)) < 1$ and given any $\varepsilon > 0$, find a large subset $B \subset X \setminus q(A)$ with submeasure $\mu(B) < \varepsilon$. Then the set $B' = q^{-1}(B) \subset G \setminus A$ is large in G and has submeasure $\mu'(B') = \mu(B) < \varepsilon$.

2. The second statement is a partial case of the first one.

3. Assume that G admits a homomorphism $h : G \rightarrow H$ onto an infinite totally bounded topological group H . Then the Raikov completion \bar{H} of H is a compact topological group, which carries a left-invariant probability Borel measure λ . This measure induces a syndetic submeasure $\mu(A) = \lambda(\bar{A})$, $A \subset H$, on the group H and the syndetic submeasure $\nu(A) = \mu(q(A))$, $A \subset G$, on the group G .

4. If G is residually finite, then G admits an injective homomorphism into a product $\prod_{\alpha \in A} G_\alpha$ of finite groups, which is a compact topological group with respect to the Tychonoff product topology. Now it suffices to apply the preceding statement.

5. Suppose that the group G contains a normal infinite amenable subgroup H . Denote by $P_\omega(H)$ the set of finitely supported probability measures on H . Each measure $\mu \in P_\omega(H)$ can be written as a convex combination $\mu = \sum_{i=1}^n \alpha_i \delta_{x_i}$ of Dirac measures concentrated at points x_i of H . We claim that the function

$$\sigma_H : \mathcal{P}(G) \rightarrow [0, 1], \quad \sigma_H : A \mapsto \inf_{\mu \in P_\omega(H)} \sup_{y \in G} \mu(Ay),$$

is a syndetic left-invariant submeasure on G .

First we prove that σ_H is left-invariant. Given any $x \in G$ and $A \subset G$ it suffices to check that $\sigma_H(xA) \leq \sigma_H(A) + \varepsilon$ for every $\varepsilon > 0$. The definition of σ_H guarantees that σ_H is right-invariant. Consequently, $\sigma_H(xA) = \sigma_H(xAx^{-1})$. By the definition of $\sigma_H(A)$, there is a finitely supported probability measure $\mu \in P_\omega(H)$ such that $\sup_{y \in G} \mu(Ay) < \sigma_H(A) + \varepsilon$. Write μ as a convex combination $\mu = \sum_{i=1}^n \alpha_i \delta_{a_i}$ of Dirac measures concentrated at points $a_1, \dots, a_n \in H$. Since H is a normal subgroup of G , the probability measure $\mu' = \sum_{i=1}^n \alpha_i \delta_{xa_i x^{-1}}$ belongs to $P_\omega(H)$. Taking into account that for every $y \in G$

$$\mu'(xAx^{-1}y) = \mu'(xAx^{-1}yxx^{-1}) = \mu(Ax^{-1}yx),$$

we conclude that

$$\sigma_H(xAx^{-1}) \leq \sup_{y \in G} \mu'(xAx^{-1}y) \leq \sup_{y \in G} \mu(Ax^{-1}yx) < \sigma_H(A) + \varepsilon.$$

So, σ_H is left-invariant.

Next, we prove that σ_H is subadditive. Given two subsets $A, B \subset G$, it suffices to check that $\sigma_H(A \cup B) \leq \sigma_H(A) + \sigma_H(B) + 3\varepsilon$ for every $\varepsilon > 0$. By the definition of the numbers $\sigma_H(A)$ and $\sigma_H(B)$, there are finitely supported probability measures $\mu_A, \mu_B \in P_\omega(H)$ such that $\sup_{y \in G} \mu_A(Ay) < \sigma_H(A) + \varepsilon$ and $\sup_{y \in G} \mu_B(By) < \sigma_H(B) + \varepsilon$. By Emerson's characterization of amenability [5, 1.7], for the probability measures μ_A and μ_B there are probability measures $\mu'_A, \mu'_B \in P_\omega(H)$ such that $\|\mu_A * \mu'_A - \mu_B * \mu'_B\| < \varepsilon$. Write the measures μ_A, μ_B, μ'_A and μ'_B as convex combinations of Dirac measures:

$$\mu_A = \sum_i \alpha_i \delta_{x_i}, \quad \mu'_A = \sum_j \alpha'_j \delta_{x'_j}, \quad \mu_B = \sum_i \beta_i \delta_{y_i}, \quad \mu'_B = \sum_j \beta'_j \delta_{y'_j}.$$

Then $\mu_A * \mu'_A = \sum_{i,j} \alpha_i \alpha'_j \delta_{x_i x'_j}$ and $\mu_B * \mu'_B = \sum_{i,j} \beta_i \beta'_j \delta_{y_i y'_j}$. For every $y \in G$ we get

$$\begin{aligned} \mu_A * \mu'_A(Ay) &= \sum_{i,j} \alpha_i \alpha'_j \delta_{x_i x'_j}(Ay) = \sum_j \alpha'_j \sum_i \alpha_i \delta_{x_i}(Ay(x'_j)^{-1}) = \\ &= \sum_j \alpha'_j \mu_A(Ay(x'_j)^{-1}) \leq \sum_j \alpha'_j \sup_{z \in G} \mu_A(Az) = \sup_{z \in G} \mu_A(Az) < \sigma_H(A) + \varepsilon. \end{aligned}$$

By analogy we can prove that $\mu_B * \mu'_B(By) \leq \sigma_H(B) + \varepsilon$. Now consider the measure $\nu = \mu_A * \mu'_A$ and observe that for every $y \in B$ we get

$$\nu(By) = \mu_A * \mu'_A(By) \leq \mu_B * \mu'_B(By) + \|\mu_A * \mu'_A - \mu_B * \mu'_B\| < \sigma_H(B) + \varepsilon + \varepsilon.$$

Then

$$\sigma_H(A \cup B) \leq \sup_{y \in G} \nu((A \cup B)y) \leq \sup_{y \in G} \nu(Ay) + \sup_{y \in G} \nu(By) < \sigma_H(A) + \varepsilon + \sigma_H(B) + 2\varepsilon = \sigma_H(A) + \sigma_H(B) + 3\varepsilon,$$

which proves the subadditivity of σ_H .

Finally we prove that the left-invariant submeasure σ_H on G is syndetic. Fix a subset $A \subset G$ of submeasure $\sigma_H(A) < 1$ and take an arbitrary $\varepsilon > 0$. Since $\sigma_H(A) < 1$, there is a finitely supported measure $\mu \in P_\omega(H)$ such that $\sup_{y \in G} \mu(Ay) < 1$. Write μ as the convex combination $\mu = \sum_{i=1}^n \alpha_i \delta_{x_i}$ of Dirac measures. We can assume that each coefficient α_i is positive. Then the finite set $F = \{x_1, \dots, x_n\}$ coincides with the support $\text{supp}(\mu)$ of the measure μ .

It follows that for every $y \in G$ we get $\mu(Ay) < 1$ and hence $F = \text{supp}(\mu) \not\subset Ay$. This ensures that the set Fy^{-1} meets the complement $X \setminus A$ and hence $y^{-1} \in F^{-1}(G \setminus A)$. So, $G = F^{-1}(G \setminus A)$ and the set $X \setminus A$ is large in G . Now take any finite subset $E \subset H$ of cardinality $|E| > 1/\varepsilon$. Using Zorn's Lemma, choose a maximal subset $B \subset G \setminus A$ which is E -separated in the sense that $Ex \cap Ey = \emptyset$ for any distinct points $x, y \in B$. The maximality of the set B guarantees that for each $x \in G \setminus A$ the set Ex meets EB , which implies that $G \setminus A \subset E^{-1}EB$ and $G = F^{-1}(G \setminus A) = F^{-1}E^{-1}EB$. This means that the set B is large in G . We claim that

$|E^{-1} \cap By| \leq 1$ for each $y \in G$. Assume conversely that $E^{-1} \cap By$ contains two distinct points by and $b'y$ with $b, b' \in B$. Then $b'b^{-1} = b'y(by)^{-1} \in E^{-1}E$ and hence $Eb' \cap Eb \neq \emptyset$, which is not possible as B is E -separated. Now consider the uniformly distributed probability measure $\nu = \frac{1}{|E|} \sum_{x \in E^{-1}} \delta_x \in P_\omega(H)$ and observe that $\sigma_H(B) \leq \sup_{y \in G} \nu(By) \leq \frac{|E^{-1} \cap By|}{|E|} \leq \frac{1}{|E|} < \varepsilon$, which means that the submeasure σ_H is syndetic. \square

Remark 3.2. For any an infinite amenable group G the syndetic submeasure σ_G witnessing that G is subamenable coincides with the right Solecki submeasure σ^R introduced in [20] and studied in [2]. For basic information on amenable group we refer the reader to [11].

4. PRETHICK SETS AND PARTITIONS OF ABELIAN GROUPS

Now we return to the problem of partition of groups into pieces which are not (k, ω) -prethick. Applying Theorem 2.2 to subamenable groups, we get:

Corollary 4.1. *Each countable subamenable group G for every $k \in \mathbb{N}$ can be covered by two subsets which are not (k, ω) -prethick.*

Since each abelian group is amenable and hence subamenable, Corollary 4.1 holds for countable abelian group. However in this case the assumption of countability is superfluous.

Theorem 4.2. *Each infinite abelian group G for every $k \in \mathbb{N}$ can be covered by two subsets which are not (k, ω) -prethick.*

Proof. Take any infinite countable abelian subgroup $H \subset G$ and embed H into a countable abelian divisible group \tilde{H} (see [6, §24]). By Baer's Theorem [6, 21.1], the identity embedding $H \rightarrow \tilde{H}$ can be extended to a homomorphism $h : G \rightarrow \tilde{H}$. Then the image $\tilde{G} = h(G)$ is a countable abelian group. Since abelian groups are amenable [11, 0.15] and hence subamenable, we can apply Corollary 4.1 and for each $k \in \mathbb{N}$ find a partition $\tilde{G} = \tilde{A} \cup \tilde{B}$ into subsets none of which is (k, ω) -prethick. Then the sets $A = h^{-1}(\tilde{A})$ and $B = h^{-1}(\tilde{B})$ are not (k, ω) -prethick in G and $A \cup B = G$. \square

5. SOME OPEN PROBLEMS

By [18, Theorem 3.9], an infinite group G can be partitioned in two large subsets $G = A_1 \cup A_2$. Clearly, A_1, A_2 are not thick and hence not $(1, \omega)$ -thick.

Problem 5.1. *Can each group be covered by two subsets none of which is $(2, \omega)$ -thick? What is the answer if G is subamenable? countable?*

By Theorem 3.1, the class of subamenable groups contains all amenable groups and all residually finite groups. The same is true for the class of initially subamenable groups introduced by Gromov [7]. Let us recall that a group G is *initially subamenable* if for each finite subset $F \subset G$ there is an injective map $h : F \rightarrow A$ into an amenable group A such that for any points $x, y \in F$ with $xy \in F$ we get $h(xy) = h(x)h(y)$. It is clear that each subgroup of an initially subamenable group is initially subamenable. An example of a (finitely presented) countable group G which is not initially subamenable was constructed by Cornulier [4]. The Cornulier group G has an infinite abelian quotient and because of that is subamenable.

Problem 5.2. *Is each initially subamenable group subamenable? Is each sofic group subamenable?*

We recall that a group G is *sofic* if it embeds into the quotient group $\oplus_{n \in \mathbb{N}}^{l_\infty} S_n / \oplus_{n \in \mathbb{N}}^{c_0} S_n$ of the l_∞ -sum of symmetric groups S_n endowed with Hamming metrics. Sofic groups were introduced by Gromov [7] and Weiss [22], and later studied in [12], [13]. The class of sofic groups includes all initially amenable groups. It is not known if each countable group is sofic.

Problem 5.3. *Is a group G subamenable if each countable (finitely generated) subgroup of G is subamenable?*

Also the following natural problem seems to be open.

Problem 5.4. *Is each (countable) group subamenable?*

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